Heat Mass Transfer (2006) 42: 679–687
DOI 10.1007/s00231-005-0615-z

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Heat transfer characteristics of boundary-layer flows induced by continuous surfaces stretched with prescribed skin friction

Received: 26 April 2004 / Accepted: 6 January 2005 / Published online: 11 January 2006

Abstract A continuous surface stretched with velocity $u_w = u_w\ (x)$ and having the temperature distribution $T_w = T_w\ (x)$ interacts with the viscous fluid in which it is immersed both mechanically and thermally. The thermal interaction is characterized by the surface heat flux $q_w = q_w\ (x)$ and the mechanical one by the skin friction $\tau_w = \tau_w\ (x)$. In the whole previous theoretical research concerned with such processes, either (u_w and T_w) or (u_w and q_w) have been prescribed as known boundary conditions. The goal of the present paper is to initiate the investigation of the boundary layer flows induced by stretching processes for which either (τ_w and T_w) or (τ_w and q_w) are the prescribed quantities. The case of an isothermal surface stretched with constant skin friction, (τ_w = const., T_w = const. $\neq T_\infty$) is worked out in detail. The corresponding flow and heat transfer characteristics are compared to those obtained for the (well known) case of a uniformly moving isothermal surface (u_w = const., T_w = const. $\neq T_\infty$).

Keywords Stretching surfaces · Boundary layers · Prescribed skin friction · Surface heat transfer · Dependence on the Prandtl number

List of Symbols

$A(x)$	Scaling function, Eqs. 15,16
$B(x)$	Scaling function, Eqs. 15,16
b	Parameter, Eq. 32a
f	Dimensionless stream function, Eq. 13
h	Dimensionless heat transfer coefficient, Eq. 34
k	Thermal conductivity, Eq. 3
L	Reference length
m	Stretching exponent, Eqs. 15a,b
n	Temperature exponent, Eq. 15c
Pr	Prandtl number, Pr $= \upsilon / \alpha$
p	Dimensionless stretching velocity, Eq. 38
q	Heat flux, Eq. 3
Q	Average wall heat flux, Eq. 62
s	sgn $(m+1)$, Eq. 41
S	Skin friction, Eq. 37
T	Temperature
u	Dimensional longitudinal velocity, Eq. 1
U	Average longitudinal velocity, Eqs. 64
v	Dimensional transversal velocity, Eqs. 1
V	Average entrainment velocity, Eqs. 66
x	Dimensional wall coordinate
X	x/L
y	Dimensional transversal coordinate
Y	y/L

Greek symbols

α	Thermal diffusivity, Eq. 1c
β	Parameter, Eq. 32b
γ	Parameter, Eq. 32c
μ	Dynamic viscosity, Eq. 2
ν	Kinematic viscosity, $\nu = \mu / \rho$
η	Similarity variable, Eq. 13b
ψ	Dimensional steam function, Eqs. 12, 13a
τ	Shear stress, Eq. 2
θ	Dimensionless temperature, Eq. 13c

Subscripts

w	Wall conditions
∞	Conditions at infinity
I	Prescribed stretching velocity
III	Prescribed skin friction

Abbreviations

BBL	Backward boundary layer
FBL	Forward boundary layer

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1 Introduction

During its manufacturing process a stretched sheet interacts with the ambient fluid both thermally and mechanically. The thermal interaction is governed by the surface heat flux $q_w = q_w(x)$. This quantity can either be prescribed or it is the output of a process in which the surface temperature distribution $T_w = T_w(x)$ has been prescribed. The mechanical interaction of a stretching sheet with the ambient fluid is governed by the skin friction $\tau_w = \tau_w(x)$ which acts in general as a surface drag. Similarly to the surface heat flux $q_w = q_w(x)$, the skin friction $\tau_w = \tau_w(x)$ also can either be prescribed or it is the result of a process in which the stretching velocity of the surface $u_w = u_w(x)$ has been prescribed. Accordingly, for a comprehensive description of the process four different combinations of the prescribed mechanical and thermal boundary conditions must be considered. These four kind of prescriptions are

$$(u_w, T_w), \tag{I}$$

$$(u_w, q_w), \tag{II}$$

$$(\tau_w, T_w), \tag{III}$$

$$(\tau_w, q_w), \tag{IV}$$

While the boundary conditions of types I and II have comprehensively been investigated in the previous literature, to the best of our knowledge the conditions of types III and IV have not been considered until now. The goal of the present paper is to initiate the research of the heat transfer characteristics of boundary layers induced by continuous stretching surfaces subjected to boundary conditions of type III and IV. To this end, this paper considers a special case of III in detail and compares the results to the well known ones corresponding to the conditions of type I. Within this topic our main concern is to compare the heat transfer coefficient h_I of a process of type (I) in which both the stretching velocity u_w and the surface temperature T_w are prescribed constants (uniformly moving isothermal surface), to the heat transfer coefficient h_{III} of a process of type (III) in which the same constant surface temperature T_w is prescribed and the skin friction τ_w is also a prescribed constant (isothermal surface stretched with constant skin friction). Obviously, in the former case the skin friction as an output associated with the prescribed stretching velocity and in the latter case the stretching velocity as an output corresponding to the prescribed skin friction also are quantities of practical interest. It is assumed throughout in this paper that the stretching surface is impermeable and that buoyancy effects as well as the effect of viscous dissipation can be neglected.

As a consequence of the absence of buoyancy forces every one of the boundary value problems (I)–(IV) splits into an independent flow boundary value problem and a forced thermal convection problem, respectively. Owing to a formal mathematical analogy, the results of the flow boundary value problems with prescribed stretching velocity or prescribed skin friction also apply to the Darcy free convection boundary layer flows from vertical surfaces adjacent to fluid saturated porous media, with prescribed surface temperature distribution or prescribed surface heat flux, respectively. However, the present heat transfer problem has no analogue in the case of porous media.

The investigation of boundary layer flows induced by continuous stretching surfaces with prescribed stretching velocity $u_w = u_w(x)$ has been initiated by the pioneering work of Sakiadis [1]. In the seminal paper of Banks [2] a comprehensive analytical and numerical investigation of the self-similar Sakiadis flows has been presented. For later developments in this field, including detailed heat transfer investigations the references [3–13] can be consulted. Concerning the free convection boundary layer flows from vertical surfaces adjacent to fluid saturated porous media (to which a part of the results of the present paper also applies), the first results have been reported by Cheng and Minkowycz [14]. For later developments, especially for the mathematical analogy mentioned above see [15–21]. A vast material and a rich list of references in this research field of porous media is presented in the book of Nield and Bejan [22] as well as in the recent monograph of Pop and Ingham [23].

2 Basic balance equations and boundary conditions

When the buoyancy forces may be neglected, the steady velocity and thermal boundary layers induced by a continuous (in general non-isothermal) stretching surface moving through a quiescent incompressible fluid of constant temperature T_∞ are governed in the boundary layer approximation by the mass, momentum and energy balance equations (see e.g. [4–7]):

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \\ &u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2}, \\ &u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}. \end{aligned} \tag{1a, b, c}$$

The x-axis is directed along the continuous stretching surface and points from the narrow extrusion slot in toward $+\infty$. The y-axis is perpendicular to x and to the direction of the slot (z-axis). u and v are the x and y components of the velocity field, respectively (Fig. 1). In the usual manufacturing situation (shown in Fig. 1) the surface issues from the slot and gives thus rise to a "forward boundary layer" flow (FBL) which moves from the slot toward $x = +\infty$. In this case the stretching velocity $u(x, 0) \equiv u_w(x)$ is positive for any $x \geq 0$. These are the Sakiadis-type boundary layer flows [1]. In the

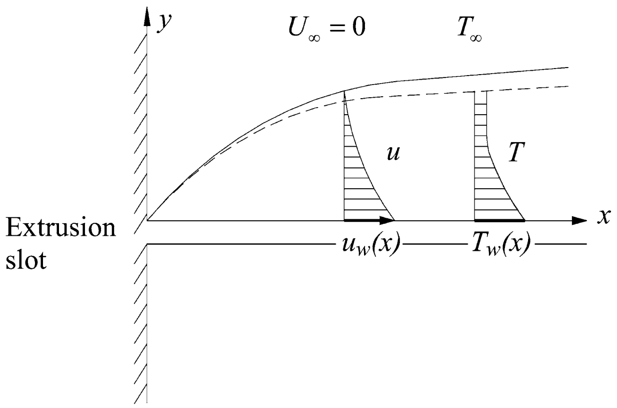


Fig. 1 Coordinate system and steady forward boundary layers induced by a stretching wall issuing from a narrow slot

opposite case in which the continuous surface coming from $x = +\infty$ enters the slot, i.e. $u(x,0) \equiv u_w(x)$ is negative for any $x \geq 0$, we are faced according to the nomenclature introduced by Goldstein [24] with the occurrence of a "backward boundary layer" flow (BBL) which moves toward the slot ($x = 0$). Such situations can be encountered in thermal treatment (e.g. annealing) of metallic sheets, wires, in the glass-fibre production, etc. A concrete example of a free convection BBL induced during the cooling of a vertically moving low-heat-resistance sheet has been investigated by Kuiken [25]. The first results concerning the free convection BBLs over cold semi-infinite vertically upwards projecting surfaces adjacent to fluid saturated porous media have recently been reported by Magyari and Keller [26].

As explained above, the process can be subjected to each two types of mechanical and thermal boundary conditions, respectively. For the momentum balance either the stretching velocity $u_w(x)$ or the skin friction

$$\tau_w(x) = \mu \frac{\partial u}{\partial y}(x,0) \tag{2}$$

can be prescribed. Similarly, for the thermal energy balance of the process, either the surface temperature distribution $T_w = T_w(x)$ or the surface heat flux

$$q_w(x) = -k \frac{\partial T}{\partial y}(x,0) \tag{3}$$

can be prescribed. Accordingly, all the four combinations (I)–(IV) of mechanical and thermal boundary conditions listed in the Introduction are of practical interest. Written down in detail these four types of boundary conditions (for impermeable surfaces) read:

I. Prescribed stretching velocity and surface temperature distribution (u_w, T_w):

$$u(x,0) = u_w(x), \quad v(x,0) = 0, \quad u(x,\infty) = 0, \tag{4a, b, c}$$

$$T(x,0) = T_w(x), \quad T(x,\infty) = T_\infty = \text{const.} \tag{5a, b}$$

II. Prescribed stretching velocity and surface heat flux (u_w, q_w):

$$u(x,0) = u_w(x), \quad v(x,0) = 0, \quad u(x,\infty) = 0, \tag{6a, b, c}$$

$$q(x,0) = q_w(x), \quad T(x,\infty) = T_\infty = \text{const.} \tag{7a, b}$$

III. Prescribed skin friction and surface temperature distribution (τ_w, T_w):

$$\tau(x,0) = \tau_w(x), \quad v(x,0) = 0, \quad u(x,\infty) = 0, \tag{8a, b, c}$$

$$T(x,0) = T_w(x), \quad T(x,\infty) = T_\infty = \text{const.} \tag{9a, b}$$

IV. Prescribed skin friction and surface heat flux (τ_w, q_w):

$$\tau(x,0) = \tau_w(x), \quad v(x,0) = 0, \quad u(x,\infty) = 0, \tag{10a, b, c}$$

$$q(x,0) = q_w(x), \quad T(x,\infty) = T_\infty = \text{const.} \tag{11a, b}$$

Equations 1 lead (for every one of the four types of boundary conditions) to an independent flow boundary value problem and a forced thermal convection problem. In terms of the stream function $\psi = \psi(x,y)$ defined by $u = \partial\psi/\partial y, v = -\partial\psi/\partial x$ Eqs. 1 reduce to

$$\begin{aligned} &\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \upsilon\frac{\partial^3\psi}{\partial y^3}, \\ &\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}. \end{aligned} \tag{12a, b}$$

3 Similarity transformations

It is well known (see e.g. Schlichting and Gersten [27]) that a transformation of the form

$$\begin{aligned} &\psi(x,y) = A(x)f(\eta), \\ &\eta = B(x)y, \\ &T(x,y) = T_\infty + C(x)\theta(\eta), \end{aligned} \tag{13a, b, c}$$

where

$$A(x) \neq \text{const.} \quad \text{and} \quad B(x) > 0 \tag{14a, b}$$

leads to two basic types of similarity solutions of Eqs. 12. These are the similarity solutions of power law type corresponding to

$$\begin{aligned} &A(x) = A_0 \cdot x^{m+1/2}, \quad B(x) = B_0 \cdot x^{m-1/2}, \quad m \neq -1, \\ &C(x) = C_0 \cdot x^n \end{aligned} \tag{15a, b, c}$$

and the similarity solutions of exponential type corresponding to

$$A(x) = A_0 \cdot e^{ax}, \quad B(x) = B_0 \cdot e^{ax}, \quad C(x) = C_0 \cdot e^{cx} \tag{16a, b, c}$$

(A_0, B_0, C_0, n, m, a and c are real constants).

The assumption $B(x) > 0$ which implies $\eta \geq 0$ can be adopted without any further restriction of generality. The assumption $A(x) \neq$ const. on the other hand, has a basic significance. Indeed, as shown recently by Magyari et al. [28], in the case $A(x) =$ const. (which for the power law similarity (15) means $m = -1$) the transformation (13a) is much to restrictive and the corresponding boundary value problem does not admit solution (neither for impermeable nor for permeable surfaces). However, if the surface is permeable and a suitable lateral suction of the fluid is admitted this "missing" boundary layer solution can readily be found by a slight extension of transformation (13a), [28].

In both of the above similarity cases the components of the corresponding velocity fields are obtained as:

$$u(x,y) = A(x)B(x)f'(\eta), \tag{17}$$

$$v(x,y) = -A'(x)\left[f(\eta) + \frac{A(x)}{A'(x)}\frac{B'(x)}{B(x)}\eta f'(\eta)\right] \tag{18}$$

(the primes denote derivatives with respect to the argument).

According to Eq. 17 the (dimensional) stretching velocity of the flow is given by

$$u_{\mathrm{w}}(x) = A(x)B(x)f'(0). \tag{19}$$

Once the assumption $B_0 > 0$ is adopted, the sign of the product $A_0 \cdot f'(0)$ decides about the sign of $u_{\mathrm{w}}(x)$ i.e. about the "forward" of "backward" character of the boundary layer considered. The condition $\{u_{\mathrm{w}}(x) > 0,\ x \geq 0\}$ of FBLs requires

$$\operatorname{sgn}(A_0) = \operatorname{sgn}[f'(0)] \quad \text{(FBLs)} \tag{20}$$

and the condition $\{u_{\mathrm{w}}(x) < 0,\ x \geq 0\}$ of BBLs requires

$$\operatorname{sgn}(A_0) = -\operatorname{sgn}[f'(0)] \quad \text{(BBLs)}. \tag{21}$$

These conditions become important in specifying the differential equations and boundary conditions satisfied by the dimensionless stream function $f = f(\eta)$ (see below).

From Eq. 18 results for the (dimensional) entrainment velocity

$$v(x,\infty) = -A'(x) \cdot f_\infty, \tag{22}$$

where

$$f_\infty = \lim_{\eta\to\infty} f(\eta) \tag{23}$$

denotes the dimensionless entrainment velocity. For the skin friction (2) and the wall heat flux (3) results

$$\tau_{\mathrm{w}}(x) = \mu A(x)B^2(x)f''(0) \tag{24}$$

and

$$q_{\mathrm{w}}(x) = -kC(x)B(x)\theta'(0) \tag{25}$$

respectively.

We restrict our further considerations to the case of power-law similarity (15) with focus on the boundary conditions (I) and (III). In this case the quantities of physical interest become:

$$\psi(x,y) = A_0 x^{m+1/2} f(\eta), \quad \eta = B_0 x^{m-1/2} y, \quad m \neq -1 \text{ and } B_0 > 0, \quad T(x,y) = T_\infty + C_0 x^n \theta(\eta), \tag{26a, b, c}$$

$$u(x,y) = A_0 B_0 x^m f'(\eta), \quad v(x,y) = -A_0 x^{m-1/2}\left[\frac{m+1}{2} f(\eta) + \frac{m-1}{2}\eta f'(\eta)\right], \tag{27a, b}$$

$$u_{\mathrm{w}}(x) = A_0 B_0 x^m f'(0), \quad v(x,\infty) = -A_0 \frac{m+1}{2} x^{m-1/2} f_\infty, \tag{28a, b}$$

$$\tau_{\mathrm{w}}(x) = \mu A_0 B_0^2 x^{3m-1/2} f''(0), \tag{29}$$

$$q_{\mathrm{w}}(x) = kB_0 C_0 x^{m+2n-1/2}\theta'(0) \quad T_{\mathrm{w}}(x) = T_\infty + C_0 x^n \theta(0). \tag{30a, b}$$

The balance equations (12) reduce to the ordinary differential equations for f and θ

$$bf''' + ff'' - \beta f'^2 = 0, \quad \frac{b}{\mathrm{Pr}}\theta'' + f\theta' - \gamma f'\theta = 0, \tag{31a, b}$$

where

$$b = \frac{2\upsilon B_0}{(m+1)A_0}, \quad \beta = \frac{2m}{m+1}, \quad \gamma = \frac{2n}{m+1} \tag{32a, b, c}$$

and $\mathrm{Pr} = \upsilon/\alpha$ is the Prandtl number.

For the boundary conditions of types (I) and (III) the value of $\theta(0)$ can be chosen equal to $+1$ without any further restriction of generality. Thus, the thermal boundary conditions of types I and III become

$$\theta(0) = 1, \quad \theta(\infty) = 0. \tag{33a, b}$$

Obviously, the thermal quantity of basic interest for both of these types of boundary conditions is the dimensionless heat transfer coefficient

$$h \equiv -\theta'(0). \tag{34}$$

Similarly, for the flow boundary conditions of types (I) and (III) we can chose, respectively, $f'(0) = +1$ and $f''(0) = -1$ without any further restriction of generality. Thus, the flow boundary conditions of types I and III become

$$f(0) = 0, \quad f'(0) = +1, \quad f'(\infty) = 0, \quad \text{Type (I)} \tag{35a, b, c}$$

and

$$f(0) = 0, \quad f''(0) = -1, \quad f'(\infty) = 0, \quad \text{Type (III)} \tag{36a, b, c}$$

respectively. Now, the mechanical quantity of basic interest is the skin friction

$$S \equiv f''(0) \quad \text{Type (I)} \tag{37}$$

(associated with the prescribed stretching velocity u_w) for the boundary conditions of type (I) and the dimensionless stretching velocity

$$p \equiv f'(0) \quad \text{Type (III)} \tag{38}$$

(associated with the prescribed skin friction τ_w) for the boundary conditions of type (III), respectively.

In order to restrict the large variety of possible solutions, we consider hereafter in this paper the FBLs only. According to Eqs. 20 and 35b, in the case of boundary conditions of type (I) this restriction implies

$$\mathrm{sgn}(A_0) = +1 \tag{39}$$

We adopt this prescription also for the boundary conditions of type (III). This implies in turn that in this case the FBLs, which we are interested in correspond (as expected) to positive values of the dimensionless stretching velocity $p = f'(0)$.

Now, choosing for the (positive) constant A_0 the value

$$A_0 = \frac{2\upsilon B_0}{|m+1|}. \tag{40}$$

Equation (32a) implies

$$b = \mathrm{sgn}(m+1) \equiv s \tag{41}$$

and the basic differential equations (31) of our boundary value problems become

$$sf''' + ff'' - \beta f'^2 = 0, \tag{42}$$

$$\frac{s}{\mathrm{Pr}}\theta'' + f\theta' - \gamma f'\theta = 0. \tag{43}$$

The flow boundary conditions of type (I) and (III) are given by Eqs. 35 and 36, respectively. The thermal boundary conditions are given in both of these cases by the same Eqs. 33. The flow boundary value problems (42), (35) and (42), (36) are mathematically equivalent to the problem of Darcy free convection boundary layer flows from vertical surfaces adjacent to fluid saturated porous media with prescribed wall temperature and prescribed wall heat flux, respectively (see e.g. [23]). In the present context of the boundary layer flows induced by stretching surfaces the first comprehensive investigation of the problem (42), (35) for $-\infty < \beta < +\infty$ and $s = +1$ has been done by Banks [2]. In the range $-2 < \beta \leq +2$ (i.e. $-1/2 < m \leq +\infty$) the solutions described by Banks [2] correspond to the usual forward boundary layers. At $\beta = -2$ (i.e. $m = -1/2$) the solution becomes singular and for $-\infty < \beta < -2$ (i.e. $-1 < m < -1/2$) no solutions exist. Banks [2] also gives numerical solutions for $s = +1$ and β-values in the range $\beta > +2$ (i.e. $m < -1$). It should be underlined, however, that these solutions do not correspond to forward but to backward boundary layers. This can easily be seen from Eq. 32a which for $m < -1$ and $b > 0$ requires $A_0 < 0$, i.e. sgn $u_w(x) = -1$. It is also worth mentioning here that in the physical context of free convection flows in saturated porous media Ingham and Brown [29] have proved rigorously that the problem (42), (35) does not admit FBL solutions in the whole range $m < -1/2$, i.e. neither for $-\infty < \beta < -2$ (i.e. $-1 < m < -1/2$) nor for $\beta > +2$ (i.e. $m < -1$).

4 Uniformly moving isothermal surface

The FBLs induced by a continuous uniformly moving ($m = 0$) isothermal ($n = 0$) surface are well known (see e.g. [2] and [3]). They are obtained as solutions of the flow and thermal boundary value problems of type I,

$$\begin{aligned} &f_I''' + f_I f_I'' = 0, \\ &f_I(0) = 0, \quad f_I'(0) = 1, \quad f_I'(\infty) = 0 \end{aligned} \tag{44}$$

and

$$\begin{aligned} &\frac{1}{\mathrm{Pr}}\theta_I'' + f_I\theta_I' = 0, \\ &\theta_I(0) = 1, \quad \theta_I(\infty) = 0, \end{aligned} \tag{45}$$

respectively.

The quantities of interest are the skin friction (37), the dimensionless entrainment velocity (23), the heat transfer coefficient (34) as well as the dimensionless velocity and temperature profiles $f'_I(\eta)$ and $\theta_I(\eta)$, respectively. Obviously, h_I and $\theta_I(\eta)$ depend on the Prandtl number Pr. The dimensionless skin friction S, entrainment velocity $f_{\infty,I}$ and the velocity profile $f'_I(\eta)$ can easily be determined by numerical integration of the problem (44). The result for S and $f_{\infty,I}$ is (see also [2])

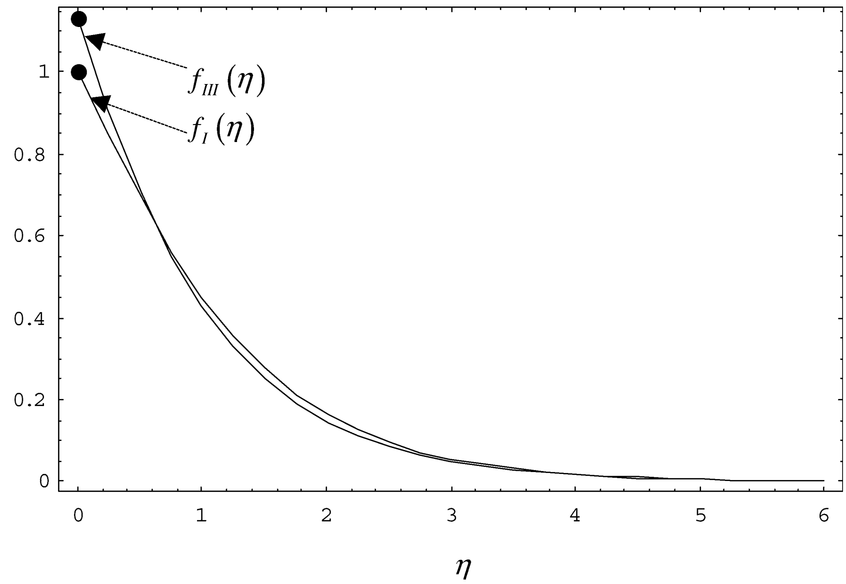
$$S = -0.62755488, \quad f_{\infty,I} = 1.14277337 \tag{46a, b}$$

The velocity profile $f'_I(\eta)$ decreases monotonically from 1 to zero as η increases from zero to infinity (Fig. 2) In terms of $f'_I(\eta)$ the skin friction S can be expressed by the integral formula

$$S = -\left(\int_0^\infty \exp\left[-\int_0^\eta f_I(\eta')\mathrm{d}\eta'\right]\mathrm{d}\eta\right)^{-1} \tag{47}$$

(which has been obtained by integrating Eq. 44 twice and taking into account of the corresponding boundary conditions).

Fig. 2 Plots of the dimensionless down stream velocities $f'(\eta)$ as functions of the similarity variable η for a surface stretched with constant velocity ($f'_I(0) = 1$, $f''_I(0) = -0.62755488$, $f_{\infty,\,\mathrm{I}} = 1.14277337$) and a surface stretched with constant skin friction ($f'_{\mathrm{III}}(0) = 1.13231319$, $f''_{\mathrm{III}}(0) = -1$, $f_{\infty,\,\mathrm{III}} = 1.13030744$), respectively



Once the flow boundary value problem (44) is solved, the solution of its thermal counterpart (45) results by quadratures

$$\theta_I(\eta) = 1 - h_I \cdot \int\limits_0^{\eta} \exp\left[-\mathrm{Pr} \cdot \int\limits_0^{\eta''} f_I(\eta')\mathrm{d}\eta'\right]\mathrm{d}\eta'', \tag{48}$$

where the heat transfer coefficient $h_I = h_I\,(\mathrm{Pr})$ is given by

$$h_I(\mathrm{Pr}) = \left(\int\limits_0^{\infty} \exp\left[-\mathrm{Pr} \cdot \int\limits_0^{\eta} f_I(\eta')\mathrm{d}\eta'\right]\mathrm{d}\eta\right)^{-1}. \tag{49}$$

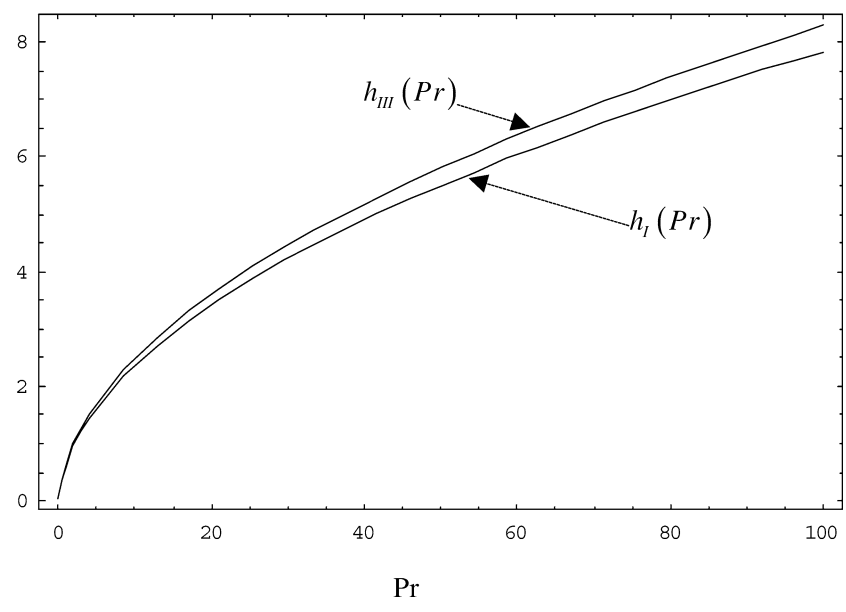
This equation shows that $h_I\,(\mathrm{Pr}) > 0$ for any value of Pr. Furthermore, comparing Eq. 49 to Eq. 47 one immediately sees that for $Pr = 1$ the heat transfer coefficient is connected to the skin friction by the simple relationship

$$h_{\mathrm{I}}(1) = -S = 0.62755488. \tag{50}$$

This result is the manifestation of the Reynolds analogy in the case of boundary layer flows induced by continuous moving surfaces, [11]. In Fig. 3 the plot of the function $h_{\mathrm{I}} = h_{\mathrm{I}}\,(Pr)$ is shown for the range $0 < \mathrm{Pr} \leq 100$ of the Prandtl number. It has been obtained from Eq. 49 with the aid of the numerical solution of the flow problem (44) for $f_{\mathrm{I}}(\eta)$.

Similarly to the velocity profile $f'_{\mathrm{I}}\,(\eta)$, temperature profiles $\theta_{\mathrm{I}}\,(\eta)$ corresponding to given values of Pr and h_{I} (Pr) also decrease monotonically from 1 to zero as η increases from zero to infinity (Fig. 4).

Fig. 3 The dimensionless heat transfer coefficients plotted as functions of the Prandtl number for a surface stretched with constant velocity, h_{I} (Pr) and a surface stretched with constant skin friction, h_{III} (Pr), respectively



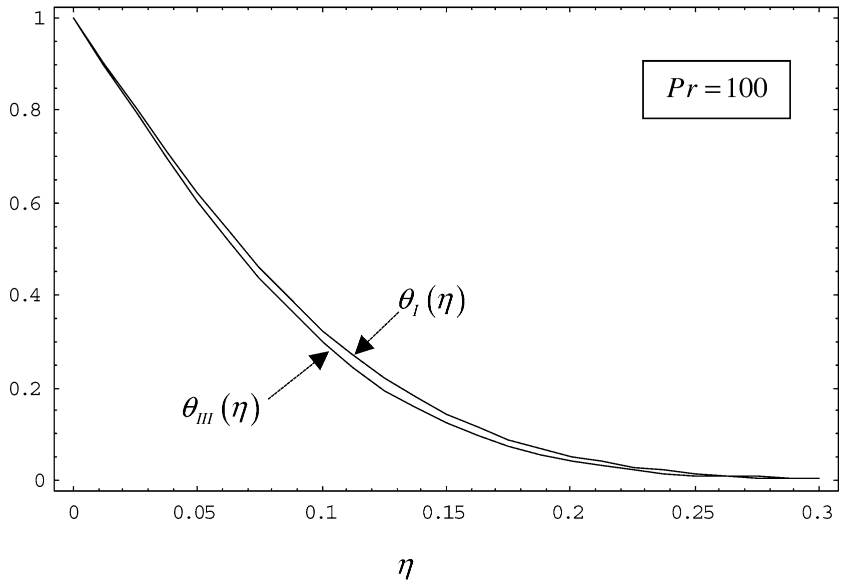


Fig. 4 Plots for Pr = 100 of the dimensionless temperature profiles $\theta(\eta)$ as functions of the similarity variable η for a surface stretched with constant velocity ($\theta_{\mathrm{I}}(\eta)$; $h_{\mathrm{I}}(100)=7.8413$) and a surface stretched with constant skin friction ($\theta_{\mathrm{III}}(\eta)$; $h_{\mathrm{III}}(100)=8.3001$), respectively

Finally, it is worth underlining that in the present case (of uniformly moving isothermal surface) neither the dimensional skin friction (29), nor the entrainment velocity (28b), nor the wall heat flux (30a) are constants. They depend on the wall coordinate x as follows

$$\tau_{\mathrm{w,I}}(x) = 2\rho\upsilon^2 B_0^3 x^{-1/2} S, \quad v_{\mathrm{I}}(x,\infty) = -\upsilon B_0 x^{-1/2} f_{\infty,\mathrm{I}}, \tag{51a, b}$$

$$q_{\mathrm{w,I}}(x) = kB_0 C_0 x^{-1/2} h_{\mathrm{I}}(\mathrm{Pr}). \tag{52}$$

The prescribed constant value of the stretching velocity (28a) in this case is $u_{\mathrm{w,I}}(x)=2\upsilon B^2_0$.

5 Isothermal surface stretched with constant skin friction

According to Eq. 29 the dimensional skin friction becomes constant for the value $m=+1/3$ of the stretching exponent. The surface being isothermal ($n=0$), the corresponding FBLs result as solutions of the flow and thermal boundary value problems

$$\begin{gathered} f'''_{\mathrm{III}} + f_{\mathrm{III}} f''_{\mathrm{III}} - \frac{1}{2} f'^2_{\mathrm{III}} = 0, \\ f_{\mathrm{III}}(0) = 0, \quad f''_{\mathrm{III}}(0) = -1, \quad f'_{\mathrm{III}}(\infty) = 0 \end{gathered} \tag{53}$$

and

$$\begin{gathered} \frac{1}{\mathrm{Pr}} \theta''_{\mathrm{III}} + f_{\mathrm{III}} \theta'_{\mathrm{III}} = 0, \\ \theta_{\mathrm{III}}(0) = 1, \quad \theta_{\mathrm{III}}(\infty) = 0, \end{gathered} \tag{54}$$

respectively.

In this case the quantities of interest are the dimensionless stretching velocity (38), the entrainment velocity (23), the heat transfer coefficient (34) as well as the dimensionless velocity and temperature profiles $f'_{\mathrm{III}}(\eta)$ and $\theta_{\mathrm{III}}(\eta)$, respectively. The dimensionless stretching velocity p, the entrainment velocity $f_{\infty,\mathrm{III}}$ and the velocity profile $f'_{\mathrm{III}}(\eta)$ can easily be determined also in this case by numerical integration of the problem (51). The result for p and $f_{\infty,\mathrm{III}}$ is (see also [17])

$$p = 1.13231319, \quad f_{\infty,\mathrm{III}} = 1.13030744 \tag{55a, b}$$

The velocity profile $f'_{\mathrm{III}}(\eta)$ decreases monotonically from p to zero as η increases from zero to infinity (see Fig. 2) In terms of $f'_{\mathrm{III}}(\eta)$ the dimensionless stretching velocity p can be expressed by the integral formula

$$p = \left(5 \cdot \int_0^\infty f_{\mathrm{III}}(\eta) f'^2_{\mathrm{III}}(\eta) \mathrm{d}\eta \right)^{+1/2}. \tag{56}$$

Once the flow boundary value problem (53) is solved, the solution of its thermal counterpart (54) results also in this case by quadratures,

$$\theta_{\mathrm{III}}(\eta) = 1 - h_{\mathrm{III}} \cdot \int_0^\eta \exp\left[-\mathrm{Pr} \cdot \int_0^{\eta''} f_{\mathrm{III}}(\eta') \mathrm{d}\eta' \right] \mathrm{d}\eta'', \tag{57}$$

where the heat transfer coefficient $h_{\mathrm{III}}=h_{\mathrm{III}}(\mathrm{Pr})$ is given by

$$h_{\mathrm{III}}(\mathrm{Pr}) = \left(\int_0^\infty \exp\left[-\mathrm{Pr} \cdot \int_0^\eta f_{\mathrm{III}}(\eta') \mathrm{d}\eta' \right] \mathrm{d}\eta \right)^{-1}. \tag{58}$$

This equation shows that $h_{\mathrm{III}}(\mathrm{Pr}) > 0$ for any value of Pr. The plot of the function $h_{\mathrm{III}} = h_{\mathrm{III}}(\mathrm{Pr})$ is also shown in Fig. 3 for the range $0 < \mathrm{Pr} \leq 100$ of the Prandtl number. It has been obtained from Eq. 58 with the aid of the numerical solution of the flow problem (43) for $f_{\mathrm{III}}(\eta)$. Similarly to the velocity profile $f'_{\mathrm{III}}(\eta)$, temperature profiles $\theta_{\mathrm{III}}(\eta)$ corresponding to given values of Pr and $h_{\mathrm{III}}(\mathrm{Pr})$ also decrease monotonically from 1 to zero as η increases from zero to infinity (Fig. 4).

The dimensional stretching velocity (28a), the entrainment velocity (28b), and the wall heat flux (30a) are functions of the wall coordinate x,

$$u_{\mathrm{w,III}}(x) = \frac{3}{2}\upsilon B_0^2 x^{+1/3} p, \quad v_{\mathrm{III}}(x,\infty) = -\upsilon B_0 x^{-1/3} f_{\infty,\mathrm{III}}, \tag{60a, b}$$

$$q_{\mathrm{w,III}}(x) = k B_0 C_0 x^{-1/3} h_{\mathrm{III}}(\mathrm{Pr}). \tag{61}$$

The prescribed constant value of the skin friction (29) in this case is $\tau_{\mathrm{w,III}}(x) = -3\rho\upsilon^2 B_0^3$.

6 Discussion

The dimensionless quantities labeled by I and III and plotted in Figs. 2, 3, 4 would suggest at a first sight that the boundary layer flows induced by continuous isothermal surfaces ($T_{\mathrm{w}} = \mathrm{const.} \neq T_\infty$) which are stretched with constant velocity (label I) and constant skin friction (label III), respectively, do not differ substantially from each other. Such an interpretation however is physically misleading since it does not take into account that (i) in Figs. 2 and 4 the dimensionless velocity and temperature profiles I and III are compared to each other on different scales of the physical coordinates x and y, and (ii) the dimensional velocities, $u_{\mathrm{I}}(x,y)$ and $u_{\mathrm{III}}(x,y)$ as well as the dimensional wall heat fluxes $q_{\mathrm{w,I}}(x)$ and $q_{\mathrm{w,III}}(x)$ scale with the wall coordinate differently. Accordingly the dimensional quantities which are "seen" in an actual stretching process will look quite differently. This is easily seen by comparing Eqs. 26–30 taken for values $m = 0$ and $m = +1/3$ which correspond to the cases I and III, respectively. Furthermore, the average wall heat flux through a part of length L of the stretching surface, expressed in units of kC_0/L,

$$Q = \int_0^1 \frac{q_{\mathrm{w}}(x)}{(kC_0/L)} \mathrm{d}X \tag{62}$$

is given in the two cases by equations

$$Q_I(\mathrm{Pr}) = 2h_{\mathrm{I}}(\mathrm{Pr}) \quad \text{and} \quad Q_{\mathrm{III}}(\mathrm{Pr}) = \frac{3}{2}h_{\mathrm{III}}(\mathrm{Pr}), \tag{63a, b}$$

respectively. Similarly, it is also interesting to compare the average stretching velocities (expressed in units of $2\upsilon/L$),

$$U_{\mathrm{w,I}} = 1 \quad \text{vs.} \quad U_{\mathrm{w,III}} = \frac{9p}{16} = 0.6369 \tag{64a, b}$$

the average skin frictions (expressed in units of $3\rho\upsilon^2/2L^2$),

$$\bar{\tau}_{\mathrm{w,I}} = -\frac{8|S|}{3} = -1.6735 \quad \text{vs.} \quad \bar{\tau}_{\mathrm{w,III}} = -1 \tag{65a, b}$$

as well as the average entrainment velocities (expressed in units of $2\upsilon/L$),

$$V_{\mathrm{I}}(\infty) = -f_{\infty,\mathrm{I}} = -1.1427 \quad \text{vs.} \quad V_{\mathrm{III}}(\infty) = -\frac{3}{4}f_{\infty,\mathrm{III}} = -0.8477. \tag{66a, b}$$

While according to Fig. 3 $h_{\mathrm{I}}(\mathrm{Pr}) < h_{\mathrm{III}}(\mathrm{Pr})$, Fig. 5 shows that due to the different X scales of the wall heat fluxes $q_{\mathrm{w,I}}(x)$ and $q_{\mathrm{w,III}}(x)$, for the average wall heat fluxes $Q_{\mathrm{I}}(\mathrm{Pr})$ and $Q_{\mathrm{III}}(\mathrm{Pr})$ given by Eqs. 62 the converse inequality $Q_{\mathrm{I}}(\mathrm{Pr}) > Q_{\mathrm{III}}(\mathrm{Pr})$ holds. Obviously, the actual physical situation is described by the curves shown in

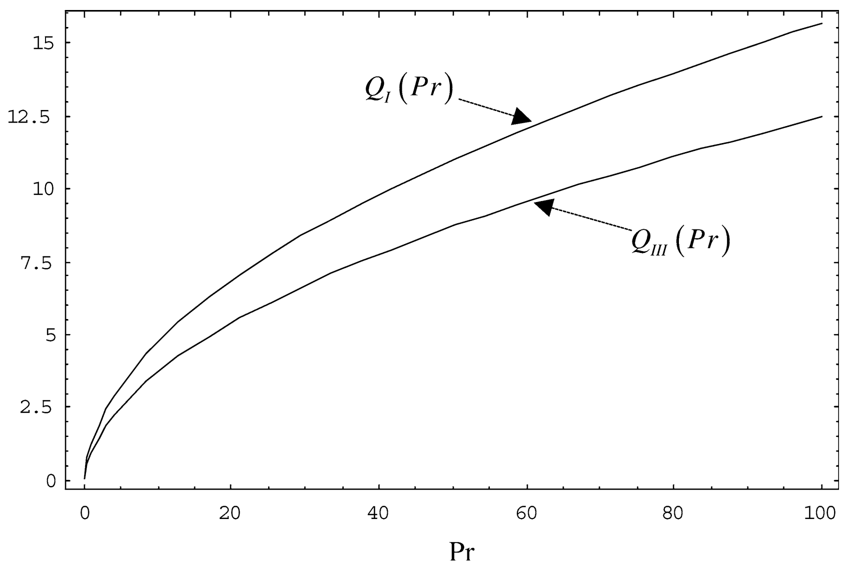


Fig. 5 The average wall heat fluxes given by Eqs. 63 plotted as functions of the Prandtl number for a surface stretched with constant velocity, $Q_{\mathrm{I}}(\mathrm{Pr})$, and a surface stretched with constant skin friction, $Q_{\mathrm{III}}(\mathrm{Pr})$, respectively

Fig. 5. These results yield in fact the main motivation for a further investigation of the flows induced by surfaces stretching with a prescribed skin friction.

7 Summary and conclusions

In this paper the mechanical and thermal characteristics of the boundary layer flows induced by continuous isothermal surfaces stretched with constant skin friction (label III) have been compared to those of the well-known boundary layer flows induced by surfaces stretched with constant velocity (label I). Compared on the different scales of the two similarity variables η, the similar velocity profiles f'_{I} and f'_{III} as well as the similar temperature profiles θ_{I} and θ_{III} deviate only slightly from each other. The same holds for the heat transfer coefficients h_{I} (Pr) and h_{III} (Pr) (see Figs.2, 3, 4). However, when compared on the actual physical scales of the wall coordinate X and the transversal coordinate Y, the velocity profiles u_{I} and u_{III}, the temperature profiles θ_{I} and θ_{III} as well as the wall heat fluxes $q_{\mathrm{w,I}}$ and $q_{\mathrm{w,III}}$, respectively, show in general substantial deviations from each other. The same holds, especially for large values of Pr for the average wall heat fluxes Q_{I} and Q_{III} (see Fig. 5). Since only the case of the constant prescribed skin friction has been discussed above, and in addition, the buoyancy as well as the effect of viscous dissipation has been neglected, the present paper leaves open several opportunities of practical interest for a future research in this field.

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